Approximation of Pi according to the method of Archimedes

(in the form of Christian Wolff)

Archimedes of Syracuse, approx. 287-212 BC
Christian Freiherr von Wolff, approx. 1679-1754 AD

Reference: Ziegenbalg: Algorithmen von Hammurapi bis Gödel, 4. Auflage, Springer-Spektrum, Wiesbaden 2016, section 3.2.3

1 Using symbolic and rational arithmetic

```
(%i1) verbose : true;          /*   control variable for printout */
(verbose) true
```

```
(%i2) Pi_Archimedes_Wolff(s) :=
  block([r:1, se, su, ue, uu, i, n:3],
    se : sqrt(3),          /*  initial values  */
    ue : 3 * se,           /*  start with triangle  */
    su : 2 * sqrt(3),
    uu : 3 * su,
    if verbose then print(n, "   ", ue/2, "   ", uu/2, "   ", se*se),
    for i : 1 step 1 thru s do
      (n  : n * 2,
       se : fullratsimp(r*sqrt(2-2*sqrt(1-(se/(2*r))*(se/(2*r))))),
       ue : n * se,
       su : fullratsimp(se / sqrt(1 - (se/(2*r)) * (se/(2*r)) ) ),
       uu : n * su,
       if verbose then print(i, n, ue/2, uu/2, se*se) ),
    (ue/2+uu/2)/2 ) $ ;
```

(%i3) Pi_Archimedes_Wolff(5);

$$3 \quad \frac{3^{3/2}}{2} \quad 3^{3/2} \quad 3$$

$$1 \quad 6 \quad 3 \quad 2\sqrt{3} \quad 1$$

$$2 \quad 12 \quad 6\sqrt{2-\sqrt{3}} \quad \frac{12\sqrt{2-\sqrt{3}}}{\sqrt{\sqrt{3}+2}} \quad 2-\sqrt{3}$$

$$3 \quad 24 \quad 12\sqrt{2-\sqrt{\sqrt{3}+2}} \quad \frac{24\sqrt{2-\sqrt{\sqrt{3}+2}}}{\sqrt{\sqrt{\sqrt{3}+2}+2}} \quad 2-\sqrt{\sqrt{3}+2}$$

$$4 \quad 48 \quad 24\sqrt{2-\sqrt{\sqrt{\sqrt{3}+2}+2}} \quad \frac{48\sqrt{2-\sqrt{\sqrt{\sqrt{3}+2}+2}}}{\sqrt{\sqrt{\sqrt{\sqrt{3}+2}+2}+2}} \quad 2-\sqrt{\sqrt{\sqrt{3}+2}+2}$$

$$5 \quad 96 \quad 48\sqrt{2-\sqrt{\sqrt{\sqrt{\sqrt{3}+2}+2}+2}} \quad \frac{96\sqrt{2-\sqrt{\sqrt{\sqrt{\sqrt{3}+2}+2}+2}}}{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{3}+2}+2}+2}+2}} \quad 2-\sqrt{\sqrt{\sqrt{\sqrt{3}+2}+2}+2}$$

(%o3)
$$\frac{\frac{96\sqrt{2-\sqrt{\sqrt{\sqrt{\sqrt{3}+2}+2}+2}}}{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{3}+2}+2}+2}+2}}+48\sqrt{2-\sqrt{\sqrt{\sqrt{\sqrt{3}+2}+2}+2}}}{2}$$

2 Using floating point arithmetic

(%i4)
```
Pi_Archimedes_Wolff_floating_point(s) :=
      /* in this version computation is done in floating point mode */
      /* mind the "catastrophic cancellation" (in German:  "Subtraktionskatastrophe") */
 block([r:1, se, su, ue, uu, i, n:3],
   se : float(sqrt(3)),          /* initial values              */
   ue : 3 * se,                  /* for the "triangle"-polygon */
   su : 2 * float(sqrt(3)),
   uu : 3 * su,
   print( 0, n, ue/2, uu/2, se*se),
   for i : 1 step 1 thru s do
     (n  : n * 2,
      se : r*sqrt(2-2*sqrt(1-(se/(2*r))*(se/(2*r)))),
      ue : n * se,
      su : se / sqrt(1 - (se/(2*r)) * (se/(2*r)) ),
      uu : n * su,
      printf(true, "~2d ~10d ~23,20h ~23,20h ~43,40h ~%",
                   i, n, ue/2, uu/2, se*se) ),
   (ue/2+uu/2)/2 ) $ ;
```

(%i5)
```
Pi_Archimedes_Wolff_floating_point(30);
```

0 3 2.598076211353316 5.196152422706632 3.0

1	6	3.000000000000000000	3.464101615137754000	0.999999999999998000000
2	12	3.105828541230250000	3.215390309173473000	0.267949192431122800000
3	24	3.132628613281237000	3.159659942097499000	0.068148347421863380000
4	48	3.139350203046872000	3.146086215131440000	0.017110277252379230000
5	96	3.141031950890530000	3.142714599645388000	0.004282153522793042000
6	192	3.141452472285344000	3.141873049979706000	0.001070825047268631000
7	384	3.141557607911622000	3.141662747056613000	0.000267724180876394100
8	768	3.141583892148936000	3.141610176605307000	0.000066932165197819400
9	1536	3.141590463236762000	3.141597034330238000	0.000016733111298794510
10	3072	3.141592106043048000	3.141593748815128000	0.000004183282199754501
11	6144	3.141592516588155000	3.141592927281095000	0.000001045820823275534
12	12288	3.141592618640789000	3.141592721314019000	0.000000261455222805293
13	24576	3.141592645321216000	3.141592670989523000	0.000000065363806811540
14	49152	3.141592645321216000	3.141592651738293000	0.000000016340951702880
15	98304	3.141592645321216000	3.141592646925485000	0.000000004085237925721
16	196608	3.141592645321216000	3.141592645722284000	0.000000001021309481430
17	393216	3.141593669849427000	3.141593669949694000	0.000000000255327536890
18	786432	3.141592303811738000	3.141592303836805000	0.000000000063831828710
19	1572864	3.141608696224804000	3.141608696231071000	0.000000000015958123710
20	3145728	3.141586839655041000	3.141586839656608000	0.000000000003989475410
21	6291456	3.141674265021758000	3.141674265022149000	0.000000000000997424300
22	12582912	3.141674265021758000	3.141674265021855000	0.000000000000249356000
23	25165824	3.143072740170040000	3.143072740170064000	0.000000000000062394500
24	50331648	3.159806164941135000	3.159806164941140000	0.000000000000015765100
25	100663296	3.181980515339464000	3.181980515339466000	0.000000000000003996800
26	201326592	3.354101966249685000	3.354101966249686000	0.000000000000001110200
27	402653184	4.242640687119286000	4.242640687119286000	0.000000000000000444000
28	805306368	6.000000000000000000	6.000000000000000000	0.000000000000000222000
29	1610612736	0.000000000000000000	0.000000000000000000	0.000000000000000000000
30	3221225472	0.000000000000000000	0.000000000000000000	0.000000000000000000000

(%o5) 0.0

3 Using bigfloat arithmetic

```
(%i6)  Pi_Archimedes_Wolff_bigfloat(s) :=
        /* in this version all computation is done in bigfloat mode */
        block([r:1, se, su, ue, uu, i, n:3],
          se : bfloat(sqrt(3)),  /* initial values            */
          ue : bfloat(3 * se),   /* for the "triangle"-polygon */
          su : bfloat(2 * sqrt(3)),
          uu : bfloat(3 * su),
          printf(true, "~2d ~10d ~23,20h ~23,20h ~43,40h ~%",
                       0, n, ue/2, uu/2, se*se),
          for i : 1 step 1 thru s do
            (n  : bfloat(n * 2),
             se : bfloat(r*sqrt(2-2*sqrt(1-(se/(2*r))*(se/(2*r))))),
             ue : bfloat(n * se),
             su : bfloat(se / sqrt(1 - (se/(2*r)) * (se/(2*r)))),
             uu : bfloat(n * su),
             printf(true, "~2d ~10d ~23,20h ~23,20h ~43,40h ~%",
                          i, n, ue/2, uu/2, se*se) ),
             /* in printf:  ~2d :  print as integer, 2 places, right-justified
                      ~23,20h :  print as decimal right-justified,
                                 23 places altogether, 20 places after the decimal point */
          bfloat(0.5*(0.5*ue+0.5*uu)) ) $ ;

(%i7)  set_display(ascii)  ;
(%o7)                          ascii

(%i8)  fpprec : 100;
(%o8)                           100
```

(%i9) Pi_Archimedes_Wolff_bigfloat(30);

```
 0       3  2.59807621135331594029  5.19615242270663188058  3.00000000000000000000000
 1       6  3.00000000000000000000  3.46410161513775458705  1.00000000000000000000000
 2      12  3.10582854123024914819  3.21539030917347247767  0.26794919243112270647253
 3      24  3.13262861328123819716  3.15965994209750048332  0.06814834742186342650051
 4      48  3.13935020304686720714  3.14608621513143497110  0.01711027725237917771088
 5      96  3.14103195089050963811  3.14271459964536829817  0.00428215352279298652380
 6     192  3.14145247228546207545  3.14187304997982387175  0.00107082504726871114030
 7     384  3.14155760791185764552  3.14166274705684852622  0.00026772418087643427450
 8     768  3.14158389214831840867  3.14161017660468953876  0.00006693216519779308470
 9    1536  3.14159046322805009574  3.14159703432152615199  0.00001673311129870170480
10    3072  3.14159210599927155054  3.14159374877135202798  0.00000418328219963791600
11    6144  3.14159251669215744759  3.14159292738509703355  0.00000104582082334477700
12   12288  3.14159261936538395519  3.14159272203861381834  0.00000026145522292590100
13   24576  3.14159264503369089667  3.14159267070199804788  0.00000006536380679958100
14   49152  3.14159265145076765170  3.14159265786784441984  0.00000001634095176665200
15   98304  3.14159265305503684169  3.14159265465930603250  0.00000000408523794583300
16  196608  3.14159265345610413926  3.14159265385717143689  0.00000000102130948671000
17  393216  3.14159265355637096366  3.14159265365663778806  0.00000000025532737169000
18  786432  3.14159265358143766976  3.14159265360650437586  0.00000000006383184292000
19 1572864  3.14159265358770434629  3.14159265359397102281  0.00000000001595796073000
20 3145728  3.14159265358927101542  3.14159265359083768455  0.00000000000398949018000
21 6291456  3.14159265358966268270  3.14159265359005434998  0.00000000000099737254000
22 12582912  3.14159265358976059952  3.14159265358985851634  0.00000000000024934310000
23 25165824  3.14159265358978507873  3.14159265358980955793  0.00000000000006233570000
24 50331648  3.14159265358979119853  3.14159265358979731833  0.00000000000001558390000
25 100663296  3.14159265358979272848  3.14159265358979425843  0.00000000000000389590000
26 201326592  3.14159265358979311097  3.14159265358979349345  0.00000000000000097390000
27 402653184  3.14159265358979320659  3.14159265358979330221  0.00000000000000024340000
28 805306368  3.14159265358979323049  3.14159265358979325440  0.00000000000000006080000
29 1610612736  3.14159265358979323647  3.14159265358979324245  0.00000000000000001520000
30 3221225472  3.14159265358979323796  3.14159265358979323946  0.00000000000000000380000
(%o9) 3.14159265358979323871165873540889325073781729847488891360714657974006251\
792022763542873596519258917870b0
```