# Figurate numbers - Figurierte Zahlen

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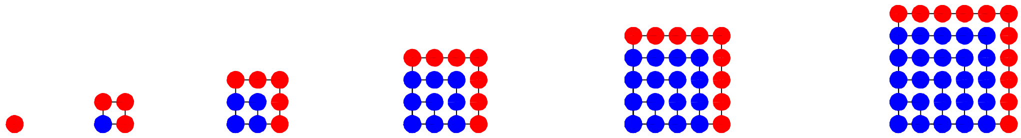
## ■ Introduction: Square numbers and triangular numbers

### ■ Square numbers

#### ■ Construction and recursive description

The most well-known figurate numbers are the *square numbers* (in German: Quadratzahlen), i.e. the numbers 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, ... .
They are called square numbers because they can be "arranged" in the shape of squares in an obvious way - and this square arrangement also explains the term "figurate number".



The red angles (i.e. the hook-like shapes) in the diagram were calles "gnomons" in ancient Greek mathematics. Each square is made up by the (blue) previous square plus a (red) gnomon. The numbers belonging to the gnomons of the squares are: 1, 3, 5, 7, 9, ... .
Since the squares' gnomons start with 1 and, step by step, increase by 2, they are identical to the odd numbers.

As the diagram shows, each square number consists of the previous square plus a suitable gnomon. Or, viewed from the other end, by starting with 1 and adding the next gnomon, we reach the next square, and so on. Since these gnomon numbers obviously are identical to the odd numbers, this shows:

**Theorem**: Each square number is the sum of consecutive odd numbers (starting with the square number 1).

**Theorem** (more precise version): Let $s$ be a square number. Then $s = 1 + 3 + 5 + ... + (2 \cdot k + 1)$ for a suitable number $k$.

*Exercise*: Describe the relation between $s$ and $k$ in the last theorem.

Let $Q_k$ be the $k$-th square number $(Q_1 = 1)$. Then, by taking a look at the pattern, we see that obviously the following equations hold

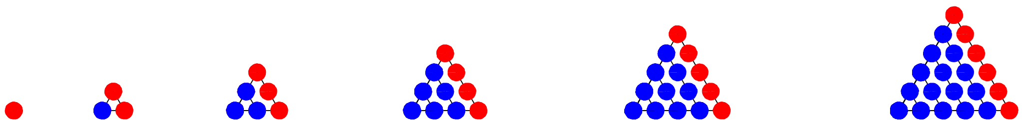
(i) $Q_k = k^2$ (this is called an "explicit" description of $Q_k$)

(ii) $Q_{k+1} = Q_k + 2 \cdot k + 1$ (this is called a "recursive" description of $Q_k$)

*Exercise*: Show that any odd square is congruent to 1 modulo 8.

## ■ Triangular numbers

The numbers 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, ... are called *triangular numbers* (in German: Dreieckszahlen). They can be represented by using trianglar patterns in the following way:



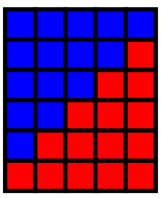
Let $T_k$ be the $k$-th triangular number $(T_1 = 1)$. Then the patterns show:

$$T_k = T_{k-1} + k$$

Expanding this equation gives

$$T_k = k + (k-1) + (k-2) + \ldots + 2 + 1 = \sum_{i=1}^{k} i$$

By the argument of the "young Gauß" (Carl Friedrich Gauß, 1777-1855), i.e. by composing triangular "stairs" appropiately,



or more formally by mathematical induction, it follows that

$$T_k = \frac{k \cdot (k+1)}{2}$$

Drawing the triangles (similarly like in Gauss' "stair" visualization above - but without the top blue row), i.e. drawing them with one right angle and two 45-degree angles, gives some insight into the relationship between triangular and square numbers: Each square is the sum of two "adjacent" triangular numbers in the following way.

**Theorem**: $Q_k = T_k + T_{k-1}$

*Proof*: Excercise (by a figurate number argument and by mathematical induction).

## ■ Polygonal numbers

### ■ Construction and recursive description

Polygonal numbers (triangular numbers, squares, pentagonal numbers, hexagonal numbers, ...) are characterized by two parameters: The number $E$ of vertices (German: Ecken) of the polygon and the stage $k$ at which it is drawn (we will always assume $E \geq 3$ and $k \geq 1$).

By $G[E, k]$ we denote the polygonal number belonging to a polygon with $E$ vertices at stage $k$. The numbers

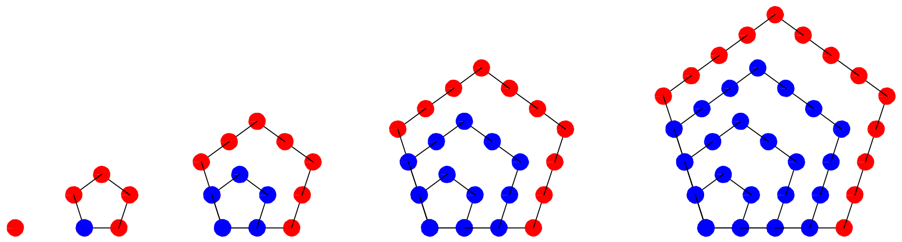
$G[3, k]$ are called triangular numbers,
$G[4, k]$ square numbers,
$G[5, k]$ pentagonal numbers,
$G[6, k]$ hexagonal numbers,
$G[7, k]$ heptagonal numbers,
$G[8, k]$ octagonal numbers,
...
$G[E, k]$ $E$-gonal numbers.

### ■ Construction of the pattern belonging to the polygonal number $G[E, k]$

Polygonal numbers are the numbers of dots in polygonal patterns in the following way: At stage $k = 1$ every polygonal pattern consits of exaxctly one dot, i.e.: $G[E, 1] = 1$. Let $k \geq 2$. The pattern belonging to $G[E, k]$ evolves out of the pattern belonging to $G[E, k-1]$ by joining an open chain of new dots to $E - 2$ sides of the old pattern so that the vertices make up a new (regular) $E$-gon with exactly $k$ dots on each of its sides.

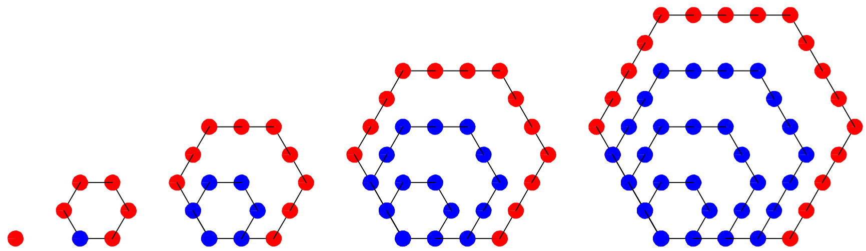
In each of the following examples the old pattern is represented by blue dots and the open chain of the new dots is represented by red dots.

*Example*: The first pentagonal patterns and numbers



1 5 12 22 35

*Example*: The first hexagonal patterns and numbers



1 6 15 28 45

From this construction the following equation follows at once:

$$G[E, k] = G[E, k-1] + (E-2) \cdot k - (E-3)$$

*Proof*: The term $G[E, k-1]$ gives the number of dots at stage $k-1$. To this, a chain of dots is added at $E-2$ sides, each side consisting of $k$ dots. This gives $(E-2) \cdot k - (E-3)$ new dots, for the dots at the $(E-3)$ "joins" belong to two sides of the new chain and must not be counted twice.

### ■ A Mathematica-Program for computing the polygonal number G[E, k]

The following (two-line) *Mathematica* program is a direct implementation of the above given description.

```
G[E_, 1] = 1 ;
G[E_, k_] := G[E, k - 1] + (E - 2) * k - (E - 3)
```

Next, we consider some uses of this program.

```
G[5, 4]
```

```
22
```

```
Table[G[5, k], {k, 1, 20}]
```

```
{1, 5, 12, 22, 35, 51, 70, 92, 117, 145,
 176, 210, 247, 287, 330, 376, 425, 477, 532, 590}
```

```
TableForm[Table[{k, G[6,  k]}, {k, 1, 20}], TableAlignments → {Right}]
```

| | |
|---|---|
| 1 | 1 |
| 2 | 6 |
| 3 | 15 |
| 4 | 28 |
| 5 | 45 |
| 6 | 66 |
| 7 | 91 |
| 8 | 120 |
| 9 | 153 |
| 10 | 190 |
| 11 | 231 |
| 12 | 276 |
| 13 | 325 |
| 14 | 378 |
| 15 | 435 |
| 16 | 496 |
| 17 | 561 |
| 18 | 630 |
| 19 | 703 |
| 20 | 780 |

```
t = Table[G[6,  k], {k, 1, 20}]
```

```
{1, 6, 15, 28, 45, 66, 91, 120, 153, 190,
 231, 276, 325, 378, 435, 496, 561, 630, 703, 780}
```

```
Apply[Plus,  t]
```

```
5530
```

## ■ Some (empirical) observations

The next table gives the first polygonal numbers from triangles to 10-gons.

```
TableForm[Table[Table[G[E,  k], {k, 1, 18}], {E, 3, 10}],
 TableAlignments -> Right,  TableSpacing → 1]
```

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 3 | 6 | 10 | 15 | 21 | 28 | 36 | 45 | 55 | 66 | 78 | 91 | 105 | 120 | 136 | 153 |
| 1 | 4 | 9 | 16 | 25 | 36 | 49 | 64 | 81 | 100 | 121 | 144 | 169 | 196 | 225 | 256 | 289 |
| 1 | 5 | 12 | 22 | 35 | 51 | 70 | 92 | 117 | 145 | 176 | 210 | 247 | 287 | 330 | 376 | 425 |
| 1 | 6 | 15 | 28 | 45 | 66 | 91 | 120 | 153 | 190 | 231 | 276 | 325 | 378 | 435 | 496 | 561 |
| 1 | 7 | 18 | 34 | 55 | 81 | 112 | 148 | 189 | 235 | 286 | 342 | 403 | 469 | 540 | 616 | 697 |
| 1 | 8 | 21 | 40 | 65 | 96 | 133 | 176 | 225 | 280 | 341 | 408 | 481 | 560 | 645 | 736 | 833 |
| 1 | 9 | 24 | 46 | 75 | 111 | 154 | 204 | 261 | 325 | 396 | 474 | 559 | 651 | 750 | 856 | 969 |
| 1 | 10 | 27 | 52 | 85 | 126 | 175 | 232 | 297 | 370 | 451 | 540 | 637 | 742 | 855 | 976 | 1105 |

The following program called **Delta[L_]** computes the differences of the adjacent numbers in any given list **L** of numbers. The program **Delta[L_, s_]** iterates this computation of differences **s** times.

```
Delta[L_] := Table[L[[i + 1]] - L[[i]], {i, 1, Length[L] - 1}];
Delta[L_,  s_] := If[s == 1,  Delta[L],  Delta[Delta[L,  s - 1]]]
```

```
Table[G[6, k], {k, 1, 20}]
```

```
{1, 6, 15, 28, 45, 66, 91, 120, 153, 190,
 231, 276, 325, 378, 435, 496, 561, 630, 703, 780}
```

```
Delta[%]
```

```
{5, 9, 13, 17, 21, 25, 29, 33, 37, 41, 45, 49, 53, 57, 61, 65, 69, 73, 77}
```

```
Delta[%]
```

```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

Applying the two-parameter **`Delta`** function from above gives the same values:

```
Delta[Table[G[6, k], {k, 1, 20}] , 2]
```

```
{4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4}
```

The next program iterates the computation of the differences until all differences are zero.

```
DiffTable[L_] :=
 Module[{T = {L}, L1},
  L1 = Delta[L];
  While[Not[Union[L1] == {0}], T = Append[T, L1]; L1 = Delta[L1]];
  T = Append[T, L1];
  Return[T] ]
```

```
TableForm[
 DiffTable[
  Table[G[6, k], {k, 1, 18}]],
 TableAlignments -> Right, TableSpacing → 1]
```

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 6 | 15 | 28 | 45 | 66 | 91 | 120 | 153 | 190 | 231 | 276 | 325 | 378 | 435 | 496 | 561 | 630 |
| 5 | 9 | 13 | 17 | 21 | 25 | 29 | 33 | 37 | 41 | 45 | 49 | 53 | 57 | 61 | 65 | 69 | |
| 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | | |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | | |

In the next example, this process is applied to all of the $E$-gonal numbers with $3 \le E \le 10$.

```
TableForm[
 Table[
  DiffTable[
   Table[G[E, k], {k, 1, 18}]], {E, 3, 10}],
 TableSpacing → 2]
```

| | | | |
|---|---|---|---|
| 1 | 2 | 1 | 0 |
| 3 | 3 | 1 | 0 |
| 6 | 4 | 1 | 0 |
| 10 | 5 | 1 | 0 |
| 15 | 6 | 1 | 0 |
| 21 | 7 | 1 | 0 |
| 28 | 8 | 1 | 0 |
| 36 | 9 | 1 | 0 |
| 45 | 10 | 1 | 0 |
| 55 | 11 | 1 | 0 |
| 66 | 12 | 1 | 0 |
| 78 | 13 | 1 | 0 |
| 91 | 14 | 1 | 0 |
| 105 | 15 | 1 | 0 |
| 120 | 16 | 1 | 0 |
| 136 | 17 | 1 | |
| 153 | 18 | | |
| 171 | | | |

| | | | |
|---|---|---|---|
| 1 | 3 | 2 | 0 |
| 4 | 5 | 2 | 0 |
| 9 | 7 | 2 | 0 |
| 16 | 9 | 2 | 0 |
| 25 | 11 | 2 | 0 |
| 36 | 13 | 2 | 0 |
| 49 | 15 | 2 | 0 |
| 64 | 17 | 2 | 0 |
| 81 | 19 | 2 | 0 |
| 100 | 21 | 2 | 0 |
| 121 | 23 | 2 | 0 |
| 144 | 25 | 2 | 0 |
| 169 | 27 | 2 | 0 |
| 196 | 29 | 2 | 0 |
| 225 | 31 | 2 | 0 |
| 256 | 33 | 2 | |
| 289 | 35 | | |
| 324 | | | |

| | | | |
|---|---|---|---|
| 1 | 4 | 3 | 0 |
| 5 | 7 | 3 | 0 |
| 12 | 10 | 3 | 0 |
| 22 | 13 | 3 | 0 |
| 35 | 16 | 3 | 0 |
| 51 | 19 | 3 | 0 |
| 70 | 22 | 3 | 0 |
| 92 | 25 | 3 | 0 |
| 117 | 28 | 3 | 0 |
| 145 | 31 | 3 | 0 |
| 176 | 34 | 3 | 0 |
| 210 | 37 | 3 | 0 |
| 247 | 40 | 3 | 0 |
| 287 | 43 | 3 | 0 |
| 330 | 46 | 3 | 0 |
| 376 | 49 | 3 | |
| 425 | 52 | | |
| 477 | | | |

| | | | |
|---|---|---|---|
| 1 | 5 | 4 | 0 |
| 6 | 9 | 4 | 0 |
| 15 | 13 | 4 | 0 |
| 28 | 17 | 4 | 0 |
| 45 | 21 | 4 | 0 |
| 66 | 25 | 4 | 0 |
| 91 | 29 | 4 | 0 |
| 120 | 33 | 4 | 0 |
| 153 | 37 | 4 | 0 |
| 190 | 41 | 4 | 0 |
| 231 | 45 | 4 | 0 |
| 276 | 49 | 4 | 0 |
| 325 | 53 | 4 | 0 |
| 378 | 57 | 4 | 0 |
| 435 | 61 | 4 | 0 |
| 496 | 65 | 4 | |
| 561 | 69 | | |
| 630 | | | |

| | | | |
|---|---|---|---|
| 1 | 6 | 5 | 0 |
| 7 | 11 | 5 | 0 |
| 18 | 16 | 5 | 0 |
| 34 | 21 | 5 | 0 |
| 55 | 26 | 5 | 0 |
| 81 | 31 | 5 | 0 |
| 112 | 36 | 5 | 0 |
| 148 | 41 | 5 | 0 |
| 189 | 46 | 5 | 0 |
| 235 | 51 | 5 | 0 |
| 286 | 56 | 5 | 0 |
| 342 | 61 | 5 | 0 |
| 403 | 66 | 5 | 0 |
| 469 | 71 | 5 | 0 |
| 540 | 76 | 5 | 0 |
| 616 | 81 | 5 | |
| 697 | 86 | | |
| 783 | | | |

| | | | |
|---|---|---|---|
| 1 | 7 | 6 | 0 |
| 8 | 13 | 6 | 0 |
| 21 | 19 | 6 | 0 |
| 40 | 25 | 6 | 0 |
| 65 | 31 | 6 | 0 |
| 96 | 37 | 6 | 0 |
| 133 | 43 | 6 | 0 |
| 176 | 49 | 6 | 0 |
| 225 | 55 | 6 | 0 |
| 280 | 61 | 6 | 0 |
| 341 | 67 | 6 | 0 |
| 408 | 73 | 6 | 0 |
| 481 | 79 | 6 | 0 |
| 560 | 85 | 6 | 0 |
| 645 | 91 | 6 | 0 |
| 736 | 97 | 6 | |
| 833 | 103 | | |
| 936 | | | |

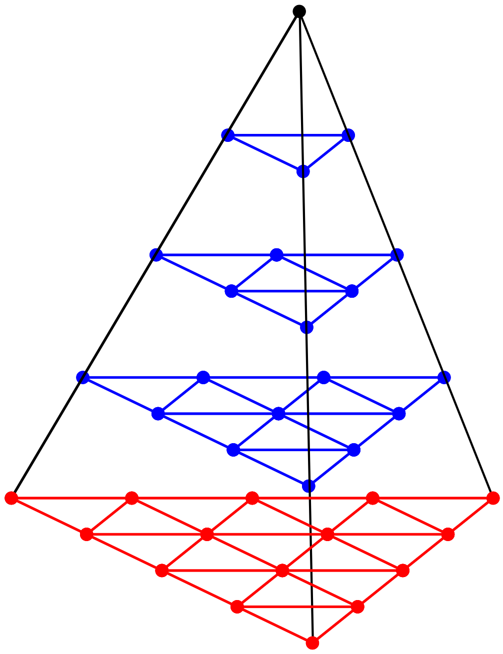
| | | | |
|---|---|---|---|
| 1 | 8 | 7 | 0 |
| 9 | 15 | 7 | 0 |
| 24 | 22 | 7 | 0 |
| 46 | 29 | 7 | 0 |
| 75 | 36 | 7 | 0 |
| 111 | 43 | 7 | 0 |
| 154 | 50 | 7 | 0 |
| 204 | 57 | 7 | 0 |
| 261 | 64 | 7 | 0 |
| 325 | 71 | 7 | 0 |
| 396 | 78 | 7 | 0 |
| 474 | 85 | 7 | 0 |
| 559 | 92 | 7 | 0 |
| 651 | 99 | 7 | 0 |
| 750 | 106 | 7 | 0 |
| 856 | 113 | 7 | |
| 969 | 120 | | |
| 1089 | | | |

| | | | |
|---|---|---|---|
| 1 | 9 | 8 | 0 |
| 10 | 17 | 8 | 0 |
| 27 | 25 | 8 | 0 |
| 52 | 33 | 8 | 0 |
| 85 | 41 | 8 | 0 |
| 126 | 49 | 8 | 0 |
| 175 | 57 | 8 | 0 |
| 232 | 65 | 8 | 0 |
| 297 | 73 | 8 | 0 |
| 370 | 81 | 8 | 0 |
| 451 | 89 | 8 | 0 |
| 540 | 97 | 8 | 0 |
| 637 | 105 | 8 | 0 |
| 742 | 113 | 8 | 0 |
| 855 | 121 | 8 | 0 |
| 976 | 129 | 8 | |
| 1105 | 137 | | |
| 1242 | | | |

■ **Closed form representations ("formulae")**

## ■ Pyramidal numbers

Pyramidal numbers (tetrahedral numbers, cubes, ...) arise from "stacking" successive polygonal numbers so as to form a pyramid.
The following picture gives a visualisation of the tetrahedral numers.



The next program obviously computes the pyramidal numbers.

```
H[E_, k_] := Sum[G[E, i], {i, 1, k}]
```

An alternative (recursive) description of the pyramidal numbers obviously is given by:

```
H2[E_, 1] = 1;
H2[E_, k_] := H2[E, k - 1] + G[E, k]
```

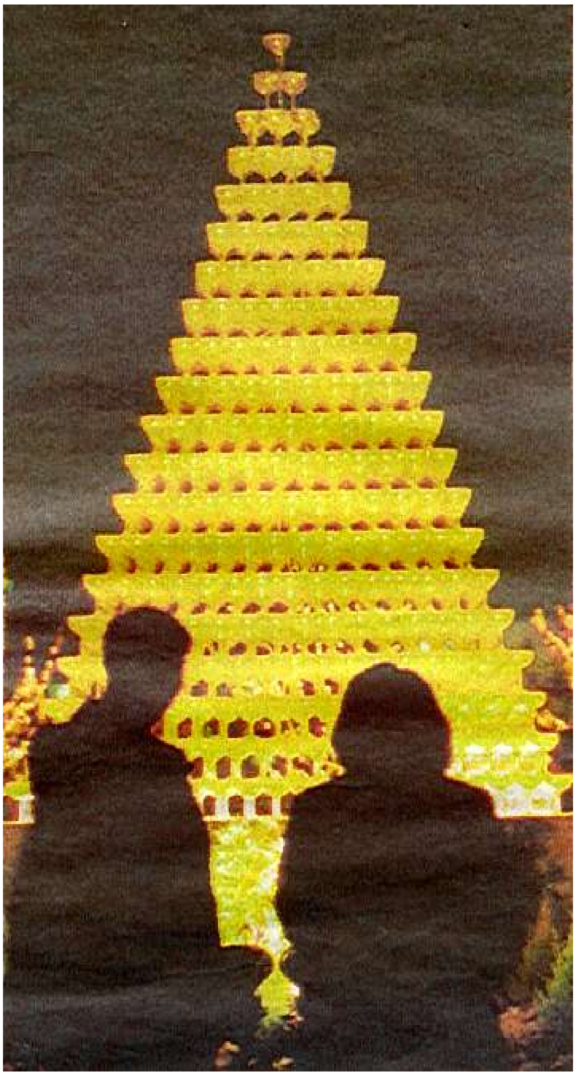
We compare some results.

```
Table[H[3, k], {k, 1, 22}]
Table[H2[3, k], {k, 1, 22}]
```

```
{1, 4, 10, 20, 35, 56, 84, 120, 165, 220, 286,
 364, 455, 560, 680, 816, 969, 1140, 1330, 1540, 1771, 2024}
```

```
{1, 4, 10, 20, 35, 56, 84, 120, 165, 220, 286,
 364, 455, 560, 680, 816, 969, 1140, 1330, 1540, 1771, 2024}
```

*Exercise*: In a newspaper article (Sonntag Aktuell, 7. Dez. 1997) it was claimed that the following Christmas tree consits of 3000 champaign glasses. Check the correctness or plausibility of this claim.



## ■ Sums of trianguar numbers, squares, n-gonal numbers

## ■ Utilities